Oxford Cambridge and RSA

Friday 22 October 2021 – Afternoon

A Level Further Mathematics A

Y545/01 Additional Pure Mathematics

Time allowed: 1 hour 30 minutes

You must have:
- the Printed Answer Booklet
- the Formulae Booklet for A Level Further Mathematics A
- a scientific or graphical calculator

INSTRUCTIONS
- Use black ink. You can use an HB pencil, but only for graphs and diagrams.
- Write your answer to each question in the space provided in the **Printed Answer Booklet**. If you need extra space use the lined pages at the end of the Printed Answer Booklet. The question numbers must be clearly shown.
- Fill in the boxes on the front of the Printed Answer Booklet.
- Answer **all** the questions.
- Where appropriate, your answer should be supported with working. Marks might be given for using a correct method, even if your answer is wrong.
- Give non-exact numerical answers correct to **3** significant figures unless a different degree of accuracy is specified in the question.
- The acceleration due to gravity is denoted by $g\,\mathrm{m\,s^{-2}}$. When a numerical value is needed use $g = 9.8$ unless a different value is specified in the question.
- Do **not** send this Question Paper for marking. Keep in the centre or recycle it.

INFORMATION
- The total mark for this paper is **75**.
- The marks for each question are shown in brackets **[ ]**.
- This document has **4** pages.

ADVICE
- Read each question carefully before you start your answer.

Answer **all** the questions.

1 **In this question you must show detailed reasoning.**

Express the number 41723_{10} in hexadecimal (base 16). **[3]**

2 The following Cayley table is for G, a group of order 6. The identity element is e and the group is generated by the elements a and b.

G	e	a	a^2	b	ab	a^2b
e	e	a	a^2	b	ab	a^2b
a	a	a^2	e	ab	a^2b	b
a^2	a^2	e	a	a^2b	b	ab
b	b	a^2b	ab	e	a^2	a
ab	ab	b	a^2b	a	e	a^2
a^2b	a^2b	ab	b	a^2	a	e

(a) List all the proper subgroups of G. **[4]**

(b) State another group of order 6 to which G is isomorphic. **[1]**

3 The points P, Q and R have position vectors $\mathbf{p} = 2\mathbf{i} + \mathbf{j} + 5\mathbf{k}$, $\mathbf{q} = \mathbf{i} - \mathbf{j} + \mathbf{k}$ and $\mathbf{r} = 2\mathbf{i} + \mathbf{j} + t\,\mathbf{k}$ respectively, relative to the origin O.

Determine the value(s) of t in each of the following cases.

(a) The line OR is parallel to $\mathbf{p} \times \mathbf{q}$. **[2]**

(b) The volume of tetrahedron $OPQR$ is 13. **[4]**

4 Solve the simultaneous linear congruences $x \equiv 1 \pmod{3}$, $x \equiv 5 \pmod{11}$, $2x \equiv 5 \pmod{17}$. **[6]**

5 The surface S has equation $x^2 + y^2 + z^2 = xyz - 1$.

(a) Show that $(2z - xy)\left(x\dfrac{\partial z}{\partial x} + y\dfrac{\partial z}{\partial y}\right) = 2(1 + z^2)$. **[6]**

(b) Deduce that S has no stationary point. **[2]**

6 The binary operation $\Diamond$ is defined on the set $\mathbb{C}$ of complex numbers by

$(a+\mathrm{i}b)\Diamond(c+\mathrm{i}d) = ac+\mathrm{i}(b+ad)$

where a, b, c and d are real numbers.

(a) Is $\mathbb{C}$ closed under $\Diamond$? Justify your answer. **[1]**

(b) Prove that $\Diamond$ is associative on $\mathbb{C}$. **[4]**

(c) Determine the identity element of $\mathbb{C}$ under $\Diamond$. **[2]**

(d) Determine the largest subset S of $\mathbb{C}$ such that (S, $\Diamond$) is a group. **[3]**

7 Let $I_n = \int_0^{\frac{1}{2}\pi} \cos^n x \, \mathrm{d}x$ for integers $n \geqslant 0$.

(a) Show that, for $n \geqslant 2$, $nI_n = (n-1)I_{n-2}$. **[4]**

(b) Use this reduction formula to deduce the exact value of I_8. **[2]**

(c) Use the results of parts **(a)** and **(b)** to determine the exact value of $\int_0^{\frac{1}{2}\pi} \cos^6 x \sin^2 x \, \mathrm{d}x$. **[2]**

8 **(a)** Solve the second-order recurrence system $H_{n+2} = 5H_{n+1} - 4H_n$ with $H_0 = 3$, $H_1 = 7$ for $n \geqslant 0$. **[5]**

(b) **(i)** Write down the quadratic residues modulo 10. **[1]**

(ii) By considering the sequence $\{H_n\}$ modulo 10, prove that H_n is never a perfect square. **[6]**

9 For each value of k the sequence of real numbers $\{u_n\}$ is given by $u_1 = 2$ and $u_{n+1} = \dfrac{k}{6+u_n}$.

For each of the following cases, either determine a value of k or prove that one does not exist.

(a) $\{u_n\}$ is constant. **[2]**

(b) $\{u_n\}$ is periodic, with period 2. **[3]**

(c) $\{u_n\}$ is periodic, with period 4. **[5]**

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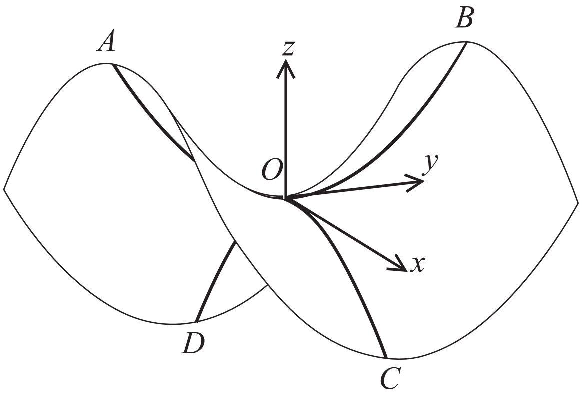

A student wishes to model the saddle of a horse. They use a surface described by a function of the form $z = \mathrm{f}(x, y)$ with a saddle point at the origin O. The z-axis is vertically upwards. The x- and y-axes lie in a horizontal plane, with the x-axis across the horse and the y-axis along the length of the horse (see diagram).

The arc AOB is part of a parabola which lies in the yz-plane. The arc COD is part of a parabola which lies in the xz-plane. The saddle is symmetric in both the xz-plane and yz-plane.

The length of the saddle, the distance AB, is to be 0.6 m with both A and B at a height of 0.27 m above O. The width of the saddle, the distance CD, is to be 0.5 m with both C and D at a depth of 0.4 m below O.

(a) On separate diagrams, sketch the sections $x = 0$ and $y = 0$. **[2]**

(b) Determine a function f that describes the saddle. [You do not need to state the domain of function f.] **[5]**

END OF QUESTION PAPER